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UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH2201**

MODULE NAME : **Algebra 3: Further Linear Algebra**

DATE : **10-May-07**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

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All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Find integers h and k such that

$$45h + 13k = 1.$$

(b) Hence solve the congruence

$$13x \equiv 2 \bmod 45.$$

(c) Find the unique $y \in \mathbb{Z}/585$ such that

$$y \equiv 3 \bmod 13 \quad \text{and} \quad y \equiv 4 \bmod 45.$$

(d) State and prove Fermat's Little Theorem.

(e) Calculate 45^{35} modulo 13.

2. (a) Let R be a commutative ring. Explain what is meant by:

(i) a unit of R;

(ii) a reducible element of R;

(iii) an irreducible element of R.

(b) (i) Find $\mathrm{hcf}(f, g)$ in $\mathbb{Q}[X]$, where $f(X) = X^3 + 2X$ and $g(X) = X^3 + 1$.

(ii) Find $a, b \in \mathbb{Q}[X]$ such that $\mathrm{hcf}(f, g) = af + bg$.

(c) Let $\mathbb{F}_3 = \{0, 1, 2\}$ denote the field with 3 elements.

(i) Find $\mathrm{hcf}(f, g)$ in $\mathbb{F}_3[X]$ when $f(X) = X^3 + 2X$ and $g(X) = X^3 + 1$.

(ii) Find $a, b \in \mathbb{F}_3[X]$ such that $\mathrm{hcf}(f, g) = af + bg$.

(iii) Factorize f and g into irreducible polynomials in $\mathbb{F}_3[X]$.

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3. Let V be a vector space over a field k and let $T : V \to V$ be a linear map.

   (a) Define the *minimal polynomial* of T.

   (b) Let $f, g \in k[X]$ satisfy $\mathrm{hcf}(f, g) = 1$. Prove that

$$\ker((f \times g)(T)) = \ker(f(T)) \oplus \ker(g(T)).$$

   (c) Define the term *generalized eigenspace.*

   (d) State and prove the *primary decomposition theorem.*

4. (a) Explain what is meant by *Jordan canonical form.*

   (b) Let $T : \mathbb{Q}^2 \to \mathbb{Q}^2$ be the linear map given by the matrix

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

   (i) Calculate the characteristic polynomial of T.

   (ii) Calculate the minimal polynomial of T.

   (iii) Find a basis for each of the generalized eigenspaces.

   (iv) Find a Jordan basis.

   (c) In each of the following cases find the Jordan normal form of the matrix A.

   (i) $\mathrm{ch}_A(X) = (X-4)^2$, $m_A(X) = (X-4)^2$.

   (ii) $\mathrm{ch}_A(X) = (X-2)(X-3)(X-4)$.

   (iii) $\mathrm{ch}_A(X) = (X-6)^5$, $m_A(X) = X-6$.

   (iv) $\mathrm{ch}_A(X) = (X-3)^8(X-5)^{10}$, $m_A(X) = (X-3)^6(X-5)^4$, $\mathrm{null}(A-3I) = 3$, $\mathrm{null}(A-5I) = 3$ and $\mathrm{null}((A-5I)^3) = 9$.

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5. (a) Let V be a finite dimensional vector space over a field k in which $1+1 \neq 0$. Let f be a symmetric bilinear form on V and let q be the associated quadratic form.

   (i) Define the orthogonal complement of a subset of V.

   (ii) Prove that if a vector $v \in V$ satisfies $q(v) \neq 0$, then

$$V = \text{span}\{v\} \oplus \{v\}^{\perp}.$$

   (iii) Prove that there is a basis for V which is orthogonal with respect to f.

   (b) Consider the following quadratic forms:

$$q_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + 2xz - y^2 + 2yz + z^2,$$

$$q_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2xz + y^2.$$

   By finding their canonical forms, determine whether q_1 and q_2 are

   (i) congruent over $\mathbb{R}$;

   (ii) congruent over $\mathbb{C}$.

6. Let V be an inner product space.

   (a) State and prove the Cauchy–Schwarz inequality for V.

   (b) Let $\{b_1, \ldots, b_n\}$ be an orthonormal basis of V. Prove that for any vector $v \in V$ we have

$$v = \langle v, b_1 \rangle b_1 + \ldots + \langle v, b_n \rangle b_n.$$

   (c) Hence show that

$$||v||^2 = |\langle v, b_1 \rangle|^2 + \ldots + |\langle v, b_n \rangle|^2.$$

   (d) Explain what is meant by an *adjoint* of a linear map $T : V \to V$.

   (e) Prove the existence and uniqueness of an adjoint of T.